Class Tutorial 1

Short review on DP - WikipediaDP

1. Rod Cutting (Knapsack variant)

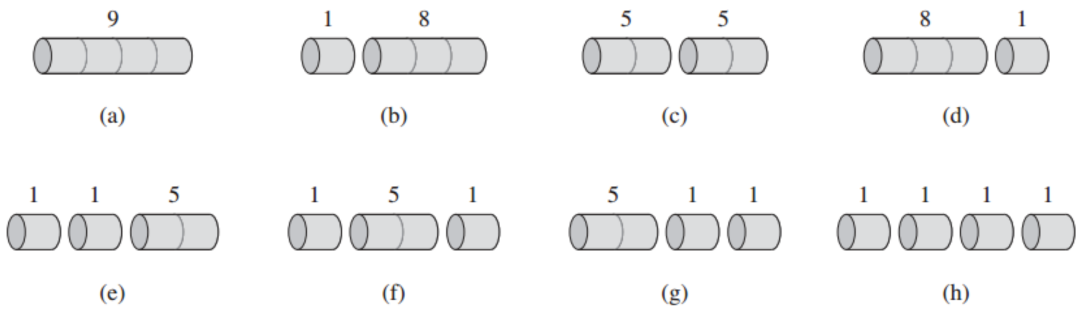
(From Introduction to Algorithms)

A company buys long steel rods and cuts them into shorter rods. The price table for the shorter rods is as follows:

Length n	1	2	3	4	5	6	7	8	9	10
Price p_n	1	5	8	9	10	17	17	20	24	30

The cost of making a cut is zero. Given a (long) rod of length n, the problem is how to cut it in order to maximize the revenue r_n.

Example: $n=4$



a. The trivial solution: enumerate all possibilities. How many different cuts exist for a rod of length n?

b. A recursive solution: Given the maximal revenues $r_1,\ldots,r_{n-1}$ compute the revenue r_n. Write down a recursive algorithm for the problem, and compute its time and space complexity.

c. Now making a cut costs c. Modify the algorithm for this case.

Solution:

a. 2^{n-1}, since each segment boundary can be cut or not.

b. $r_n = \max_{1\le i\le n}\left(p_i + r_{n-i}\right)$, since each cut can be viewed as a composition of a piece of length i, and all the other pieces.

Proof: Assume exists a revenue $\tilde{r}$ such that $\tilde{r} > r_n$. This would mean that for any $i \in \{1, .. n-1\}$ it holds that $\tilde{r} - p_i > r_{n-i}$, which is a contradiction. We assumed that for any $i \in \{1, .. n-1\}$, r_{n-i} is the maximal revenue.

The complexity time of the algorithm is $O(n^2)$ and $O(n)$ space algorithm.

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BOTTOM-UP-CUT-ROD(p, n)
let r[0..n] be a new array
r[0] = 0
for j = 1 to n
    q = −∞
    for i = 1 to j
        q = max(q, p[i] + r[j − i])
    r[j] = q
return r[n]
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c. The modified equation is $r_n = \max_{1\le i\le n}\left(p_i + r_{n-i} - c\cdot 1_{i<n}\right)$.

2. Longest Common Subsequence

(From Introduction to Algorithms)

Given a sequence $X = \langle x_1, \ldots, x_m \rangle$, we say that the sequence $Z = \langle z_1, \ldots, z_k \rangle$ is a subsequence of X if there exists a strictly increasing sequence $\langle i_1, \ldots, i_k \rangle$ such that for all $j = 1, \ldots, k$ we have $X_{i_j} = Z_j$. For example, $Z = \langle B, C, D, B \rangle$ is a subsequence of $X = \langle A, B, C, B, D, A, B \rangle$.

Given two sequences X, Y we say that Z is a *common subsequence* of X and Y if Z is a subsequence of both X and Y. In the longest-common-subsequence (LCS) problem we are given two sequences $X = \langle x_1, \ldots, x_m \rangle$ and $Y = \langle y_1, \ldots, y_n \rangle$, and we need to find the maximum length common subsequence of X and Y.

a. Warm-up: find the LCS of $X = \langle A, B, C, B, D, A, B \rangle$ and $Y = \langle B, D, C, A, B, A \rangle$.

b. Brute-force algorithm: enumeration of all subsequences. How many subsequences does X have? What is the complexity of such an algorithm?

c. Let X_i denote the i'th prefix of X : $X_i = \langle x_1, \ldots, x_i \rangle$. Prove the following theorem:

Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be sequences, and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.
2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is an LCS of X_{m-1} and Y.
3. If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is an LCS of X and Y_{n-1}.

d. Dynamic programming algorithm: let $c[i, j]$ denote the length of the LCS of X_i and Y_j. Write a recursive formula for $c[i, j]$. Derive an algorithm for the length of the LCS of X and Y. What is it's complexity?

e. (Homework) derive the actual LCS from $c[i, j]$.

Solution:

a. For example, $\langle B, C, B, A \rangle$ or $\langle B, D, A, B \rangle$.

b. 2^m

c.

Proof (1) If $z_k \neq x_m$, then we could append $x_m = y_n$ to Z to obtain a common subsequence of X and Y of length $k+1$, contradicting the supposition that Z is a *longest* common subsequence of X and Y. Thus, we must have $z_k = x_m = y_n$. Now, the prefix Z_{k-1} is a length-$(k-1)$ common subsequence of X_{m-1} and Y_{n-1}. We wish to show that it is an LCS. Suppose for the purpose of contradiction that there exists a common subsequence W of X_{m-1} and Y_{n-1} with length greater than $k-1$. Then, appending $x_m = y_n$ to W produces a common subsequence of X and Y whose length is greater than k, which is a contradiction.

(2) If $z_k \neq x_m$, then Z is a common subsequence of X_{m-1} and Y. If there were a common subsequence W of X_{m-1} and Y with length greater than k, then W would also be a common subsequence of X_m and Y, contradicting the assumption that Z is an LCS of X and Y.

(3) The proof is symmetric to (2). ■

d. Based on the previous theorem, we have

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 , \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j , \\ \max(c[i, j-1], c[i-1, j]) & \text{if } i, j > 0 \text{ and } x_i \neq y_j . \end{cases}$$

An $O(mn)$ algorithm:

```
LCS-LENGTH(X, Y)
m = X.length
n = Y.length
let b[1..m, 1..n] and c[0..m, 0..n] be new tables
for i = 1 to m
    c[i, 0] = 0
for j = 0 to n
    c[0, j] = 0
for i = 1 to m
    for j = 1 to n
        if x_i == y_j
            c[i, j] = c[i − 1, j − 1] + 1
            b[i, j] = "↖"
        elseif c[i − 1, j] ≥ c[i, j − 1]
            c[i, j] = c[i − 1, j]
            b[i, j] = "↑"
        else c[i, j] = c[i, j − 1]
            b[i, j] = "←"
return c and b
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