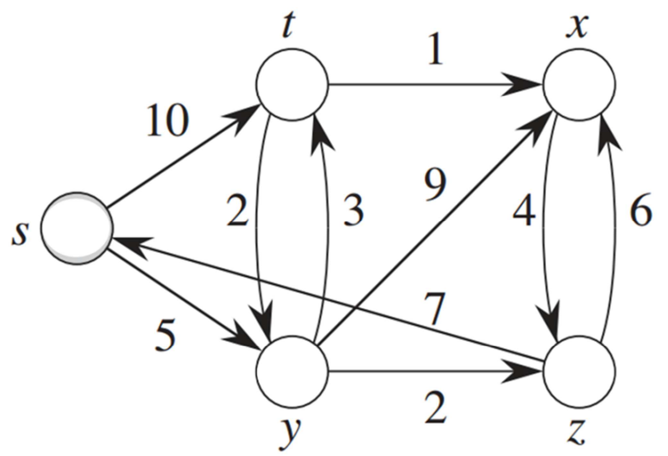
Class Tutorial 3

1. Shortest path example

Consider the following graph:



a. Run the Bellman-Ford algorithm on the graph.

b. Run Dijkstra's algorithm on the graph.

Solution

a.

Bellman-Ford Algorithm.

Input: A weighted directed graph G , and destination node t .

Initialization: $d[t]=0$, $d[v]=\infty$ for $v\in V\setminus\{t\}$, $\pi[v]=\phi$ for $v\in V$

for $i=1$ to $|V|-1$

$d'[v]=d[v], \quad v\in V\setminus\{t\}$

for each vertex $v\in V\setminus\{t\}$,

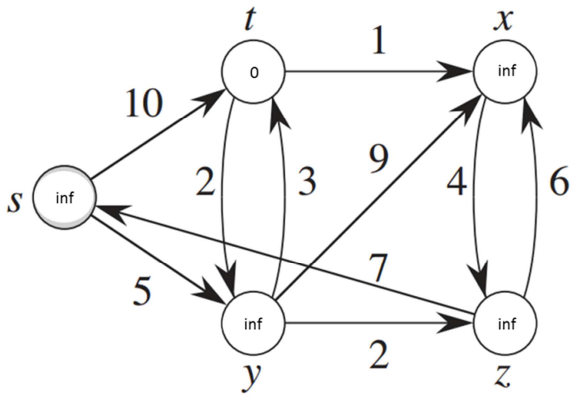
compute $q[v]=\min_u\{w(v,u)+d'[u]:(v,u)\in E\}$

if $q[v]<d[v]$,

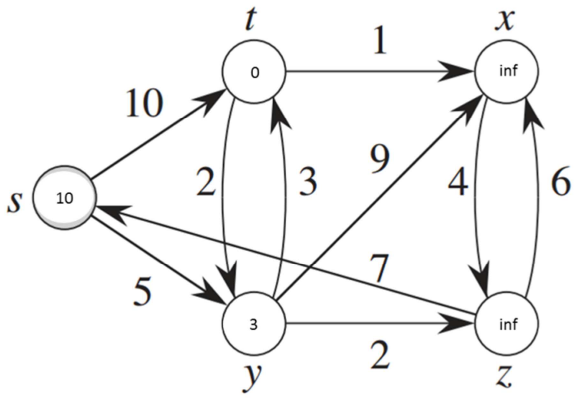
set $d[v]=q[v]$, $\pi[v]\in\arg\min_u\{w(v,u)+d'[u]:(v,u)\in E\}$

return $\{d[v],\pi[v]\}$

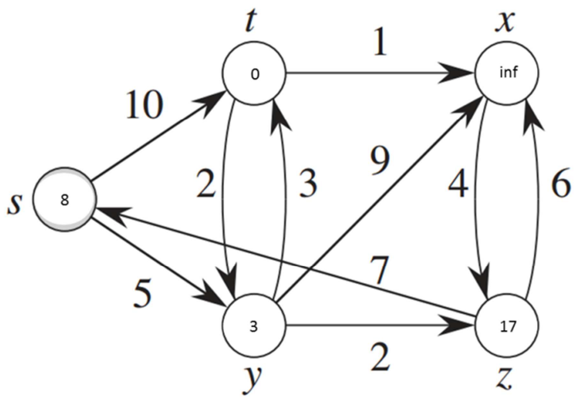
Initialization:



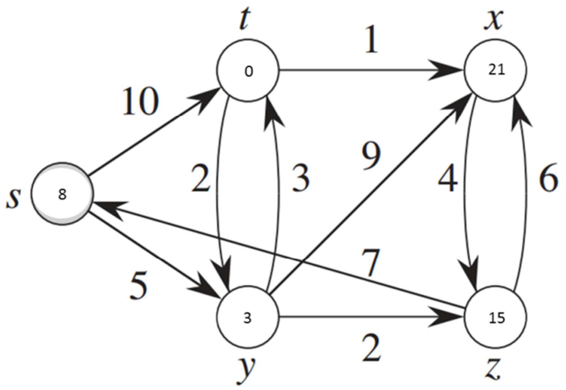
Iteration 1:



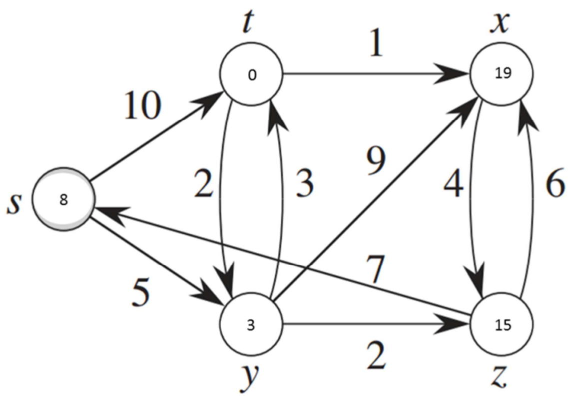
Iteration 2:



Iteration 3:



Iteration 4:



b.

Dijkstra's Algorithm.

Input: A weighted directed graph, and destination node t .

Initialization: $d[t]=0$, $d[v]=\infty$ for $v \in V \setminus \{t\}$, $\pi[v]=\phi$ for $v \in V$

$S=\phi$

while $S \neq V$, (***)

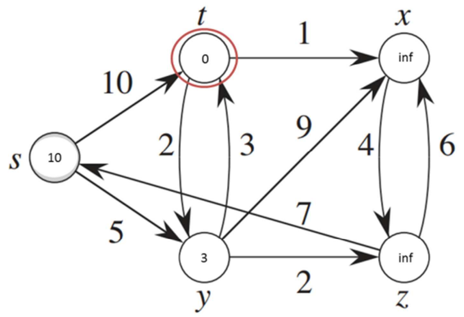
choose $u \in V \setminus S$ with minimal value $d[u]$, add it to S

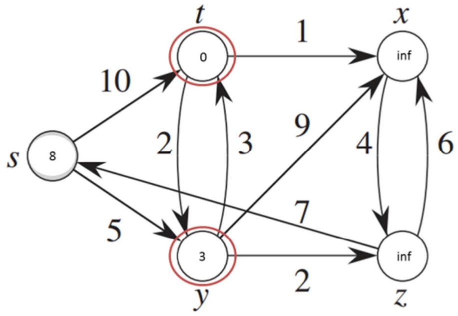
for each vertex v with $(v,u) \in E$,

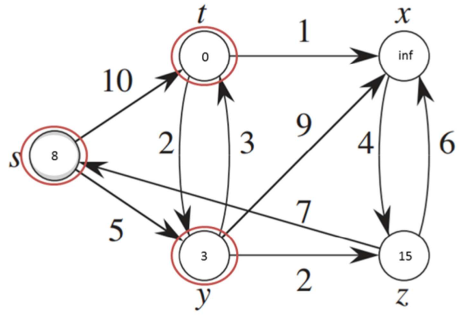
if $d[v] > w(v,u)+d[u]$,

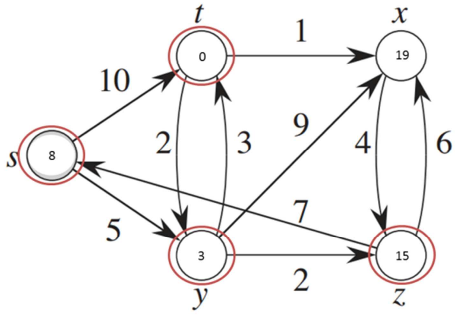
set $d[v]=w(v,u)+d[u]$, $\pi[v]=u$

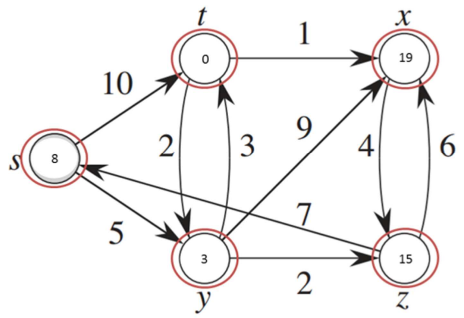
return $\{d[v],\pi[v]\}$











2. Dijkstra's Algorithm - Correctness

We let $d^*(v)$ denote the (true) shortest path length from node $v \in V$ to the destination node t, and let $d(v)$ denote the value of node v during execution on Dijkstra's algorithm. Prove the following properties:

a. Triangle inequality: for any edge $(u,v) \in E$ we have $d^*(u) \le w(u,v) + d^*(v)$.

b. Upper bound property: for all $v \in V$ and at any time in the execution of the algorithm, we have $d(v) \ge d^*(v)$. Moreover, once equality is obtained, $d(v)$ never changes.

c. Correctness of Dijkstra's algorithm: for a graph with non-negative weights, Dijkstra's algorithm terminates with $d(v) = d^*(v)$ for all $v \in V$.

Solution

a. Suppose p is a shortest path from u to t. Then p has no more weight than any other path from u to t, specifically the path that goes from u to v and then continues optimally.

b. We prove by induction on the number of relaxation steps, i.e., number of executions of

if $d[v] > w(v,u) + d[u]$,

set $d[v] = w(v,u) + d[u]$

For the first step this is clearly true, due to the initialization procedure.

Assume $d(x) \le d^*(x)$ for all $x \in V$ prior to the relaxation step, and consider a relaxation of edge (v,u). The only value that may change is $d(v)$. If it changes we have

$$\begin{aligned} d(v) &= w(v,u) + d(u) \\ &\ge w(v,u) + d^*(u) \\ &\ge d^*(v) \end{aligned}$$

Where the first inequality is by the induction hypothesis, and the second by the triangle inequality. Thus, the induction invariant is maintained.

Once $d(v) = d^*(v)$, it cannot decrease as we have now shown, and it cannot increase since relaxations only decrease values.

c. We prove the following loop invariant:

At the start of each while loop (***), we have $d(v) = d^*(v)$ for all $v \in S$.

It suffices to show that for all $u \in V$ we have $d(u) = d^*(u)$ when u is added to S. By the upper-bound property, it will never change afterwards.

Initialization: initially $S = \varnothing$ so the invariant is trivially true.

Maintenance: For the purpose of contradiction, let u be the first node added to the set S such that $d(u) \ne d^*(u)$.

We must have $u \ne t$ since t is the first node added to S and $d(t) = d^*(t) = 0$. We also have that $S \ne \varnothing$ just before u is added. There must be a path from u to t otherwise $d(u) = d^*(u) = \infty$. Thus, there is a shortest path p from u to t.

Prior to adding u to S, p connects a node in $V - S$ to a node in S. Let x denote the last node in p such that $x \in V - S$ and let y denote $x's$ successor, i.e., $y \in S$. We can decompose p into $u \to x \to y \to t$.

We claim that $d(x) = d^*(x)$ when u is added to S. To see this, observe that $d(y) = d^*(y)$ since $y \in S$ and u is the first node for which this property does not hold.

Since $x \to y \to t$ is the shortest path from x to t, when y was relaxed, we had $d(x) = w(x, y) + d^*(y) = d^*(x)$.

We now obtain the contradiction. Since x appears after u on the shortest path p, and since all weights are non-negative, we must have $d^*(x) \le d^*(u)$. Therefore

$$\begin{aligned} d(x) &= d^*(x) \\ &\le d^*(u) \\ &\le d(u) \end{aligned}$$

But because x and u were in $V - S$ we must also have $d(u) \le d(x)$, therefore, $d(x) = d^*(x) = d^*(u) = d(u)$, which contradicts our definition of u.